

# UNBOUNDED WEIGHTED CONDITIONAL TYPE OPERATORS ON $L^p(\Sigma)$

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**ABSTRACT.** In this paper we consider unbounded weighted conditional type operators on the space  $L^p(\Sigma)$ , we give some conditions under which they are densely defined and we obtain a dense subset of the domain. Also, we get that a WCT operator is continuous if and only if it is every where defined. A description of polar decomposition, spectrum and spectral radius in this context are provided. Finally, we investigate hyperexpansive WCT operators on the Hilbert space  $L^2(\Sigma)$ . As a consequence hyperexpansive multiplication operators are investigated.

## 1. Introduction

In this paper we consider a class of unbounded linear operators on  $L^p$ -spaces having the form  $M_w E M_u$ , where  $E$  is a conditional expectation operator and  $M_u$  and  $M_w$  are multiplication operators. What follows is a brief review of the operators  $E$  and multiplication operators, along with the notational conventions we will be using.

Let  $(\Omega, \Sigma, \mu)$  be a  $\sigma$ -finite measure space and let  $\mathcal{A}$  be a  $\sigma$ -subalgebra of  $\Sigma$  such that  $(\Omega, \mathcal{A}, \mu)$  is also  $\sigma$ -finite. We denote the collection of (equivalence classes modulo sets of zero measure of)  $\Sigma$ -measurable complex-valued functions on  $\Omega$  by  $L^0(\Sigma)$  and the support of a function  $f \in L^0(\Sigma)$  is defined as  $S(f) = \{t \in \Omega; f(t) \neq 0\}$ . Moreover, we set  $L^p(\Sigma) = L^p(\Omega, \Sigma, \mu)$ . We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a  $\mu$ -null set. For each  $\sigma$ -finite subalgebra  $\mathcal{A}$  of  $\Sigma$ , the conditional expectation,  $E^{\mathcal{A}}(f)$ , of  $f$  with respect to  $\mathcal{A}$  is defined whenever  $f \geq 0$  or  $f \in L^p(\Sigma)$ . In any case,  $E^{\mathcal{A}}(f)$  is the unique  $\mathcal{A}$ -measurable function for which

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on  $L^p(\Sigma)$ ,  $E^{\mathcal{A}}$  is an idempotent and  $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$ . If there is no possibility of confusion we write  $E(f)$  in place of  $E^{\mathcal{A}}(f)$  [10, 12]. This operator will play a major role in our work and we list here some of its useful properties:

- If  $g$  is  $\mathcal{A}$ -measurable, then  $E(fg) = E(f)g$ .
- $|E(f)|^p \leq E(|f|^p)$ .
- If  $f \geq 0$ , then  $E(f) \geq 0$ ; if  $f > 0$ , then  $E(f) > 0$ .
- $|E(fg)| \leq E(|f|^p)^{\frac{1}{p}} E(|g|^q)^{\frac{1}{q}}$ , (Hölder inequality) for all  $f \in L^p(\Sigma)$  and  $g \in L^q(\Sigma)$ , in which  $\frac{1}{p} + \frac{1}{q} = 1$ .

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- For each  $f \geq 0$ ,  $S(f) \subseteq S(E(f))$ .

Let  $u \in L^0(\Sigma)$ . The corresponding multiplication operator  $M_u$  on  $L^p(\Sigma)$  is defined by  $f \rightarrow uf$ . Our interest in operators of the form  $M_wEM_u$  stems from the fact that such products tend to appear often in the study of those operators related to conditional expectation. This observation was made in [1, 2, 5, 8, 9]. In this paper, first we investigate some properties of unbounded weighted conditional type operators on the space  $L^p(\Sigma)$  and then we study hyperexpansive ones.

## 2. Unbounded weighted conditional type operators

Let  $X$  stand for a Banach space and  $\mathcal{B}(X)$  for the Banach algebra of all linear operators on  $X$ . By an operator in  $X$  we understand a linear mapping  $T : \mathcal{D}(T) \subseteq X \rightarrow X$  defined on a linear subspace  $\mathcal{D}(T)$  of  $X$  which is called the domain of  $T$ . The linear map  $T$  is called densely defined if  $\mathcal{D}(T)$  is dense in  $X$  and it is called closed if its graph  $(\mathcal{G}(T))$  is closed in  $X \times X$ , where  $\mathcal{G}(T) = \{(f, Tf) : f \in \mathcal{D}(T)\}$ . We studied bounded weighted conditional type operators on  $L^p$ -spaces in [4]. Also we investigated unbounded weighted conditional type operators of the form  $EM_u$  on the Hilbert space  $L^2(\Sigma)$  in [3]. Here we investigate unbounded weighted conditional type operators of the form  $M_wEM_u$  on  $L^p$ -spaces. Let  $f$  be a positive  $\Sigma$ -measurable function on  $\Omega$ . Define the measure  $\mu_f : \Sigma \rightarrow [0, \infty]$  by

$$\mu_f(E) = \int_E f d\mu, \quad E \in \Sigma.$$

It is clear that the measure  $\mu_f$  is also  $\sigma$ -finite, since  $\mu$  is  $\sigma$ -finite. From now on we assume that  $u$  and  $w$  are conditionable (i.e.,  $E(u)$  and  $E(w)$  are defined). Operators of the form  $M_wEM_u(f) = wE(u.f)$  acting in  $L^p(\mu)$  with  $\mathcal{D}(M_wEM_u) = \{f \in L^p(\mu) : wE(u.f) \in L^p(\mu)\}$  are called weighted conditional type operators (or briefly WCT operators). In the first proposition we give a condition under which the WCT operator  $M_wEM_u$  is densely defined.

**Theorem 2.1.** *For  $1 \leq p, q < \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$  and  $E(|w|^p)^{\frac{1}{p}} E(|u|^q)^{\frac{1}{q}} < \infty$  a.e., the linear transformation  $M_wEM_u$  is densely defined.*

*Proof.* For each  $n \in \mathbb{N}$ , define

$$A_n = \{t \in \Omega : n-1 \leq E(|w|^p)(t)E(|u|^q)^{\frac{p}{q}}(t) < n\}.$$

It is clear that each  $A_n$  is  $\mathcal{A}$ -measurable and  $\Omega$  is expressible as the disjoint union of sets in the sequence  $\{A_n\}_{n=1}^\infty$ ,  $\Omega = \cup_{n=1}^\infty A_n$ .

Let  $f \in L^p(\Sigma)$  and  $\epsilon > 0$ . Then, there exists  $N > 0$  such that

$$\int_{\cup_{n=N}^\infty A_n} |f|^p d\mu = \sum_{n=N}^\infty \int_{A_n} |f|^p d\mu < \epsilon.$$

Define the sets

$$B_N = \cup_{n=N}^\infty A_n, \quad C_N = \cup_{n=1}^{N-1} A_n.$$

Then,  $\int_{B_N} |f|^p d\mu < \epsilon$  and  $C_N = \{t \in \Omega : E(|w|^p)(t)E(|u|^q)^{\frac{p}{q}}(t) < N-1\}$ . Next, we define  $g = f \cdot \chi_{C_N}$ . Clearly  $g \in L^p(\Sigma)$  and  $E(g) = E(f) \cdot \chi_{C_N}$ . Now, we show

that  $g \in \mathcal{D} = \mathcal{D}(M_w EM_u)$ . Consider the following:

$$\begin{aligned} \int_{\Omega} |wE(ug)|^p d\mu &= \int_{\Omega} |wE(uf)\chi_{C_N}|^p d\mu \\ &= \int_{C_N} E(|w|^p) |E(uf)|^p d\mu \\ &\leq \int_{C_N} E(|w|^p) E(|u|^q)^{\frac{p}{q}} |f|^p d\mu \\ &\leq (N-1) \int_{C_N} |f|^p d\mu < \infty. \end{aligned}$$

Thus,  $wE(uf) \in L^p(\Sigma)$ . Now, we show that  $\|g - f\|_p < \epsilon$ :

$$\begin{aligned} \|g - f\|_p^p &= \int_X |g - f|^p d\mu \\ &= \int_{C_N} |g - f|^p d\mu < \epsilon. \end{aligned}$$

Thus,  $\mathcal{D}$  is dense in  $L^p(\Sigma)$ .  $\square$

Here we obtain a dense subset of  $L^p(\mu)$  that we need it to proof our next results.

**Lemma 2.2.** *Let  $1 < p, q < \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $J = 1 + E(|w|^p)E(|u|^q)^{\frac{p}{q}}$ ,  $E(|w|^p)^{\frac{1}{p}}E(|u|^q)^{\frac{1}{q}} < \infty$  a.e.  $\mu$ , and  $d\nu = Jd\mu$ . We get that  $S(J) = \Omega$  and*

- (i)  $L^p(\nu) \subseteq \mathcal{D}(M_w EM_u)$ ,
- (ii)  $\overline{L^p(\nu)}^{\|\cdot\|_{\mu}} = \overline{\mathcal{D}(M_w EM_u)}^{\|\cdot\|_{\mu}} = L^p(\mu)$ .

*Proof.* Let  $f \in L^p(\nu)$ . Then

$$\|f\|_{\nu}^p d\mu \leq \|f\|_{\nu}^p < \infty,$$

so  $f \in L^p(\mu)$ . Also, by conditional-type Hölder-inequality we have

$$\begin{aligned} \|M_w EM_u(f)\|_{\nu}^p d\mu &\leq \int_{\Omega} E(|w|^p) E(|u|^q)^{\frac{p}{q}} E(|f|^p) d\mu \\ &= \int_{\Omega} E(|w|^p) E(|u|^q)^{\frac{p}{q}} |f|^p d\mu \\ &\leq \|f\|_{\nu}^p < \infty, \end{aligned}$$

this implies that  $f \in \mathcal{D}(M_w EM_u)$ . Now we prove that  $L^p(\nu)$  is dense in  $L^p(\mu)$ . By Riesz representation theorem we have

$$(L^p(\nu))^{\perp} = \{g \in L^q(\mu) : \int_{\Omega} f \cdot g d\mu = 0, \quad \forall f \in L^p(\nu)\}.$$

Suppose that  $g \in (L^p(\nu))^{\perp}$ . For  $A \in \Sigma$  we set  $A_n = \{t \in A : J(t) \leq n\}$ . It is clear that  $A_n \subseteq A_{n+1}$  and  $\Omega = \bigcup_{n=1}^{\infty} A_n$ . Also,  $\Omega$  is  $\sigma$ -finite, hence  $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$  with  $\mu(\Omega_n) < \infty$ . If we set  $B_n = A_n \cap \Omega_n$ , then  $B_n \nearrow A$  and so  $g \cdot \chi_{B_n} \nearrow g \cdot \chi_A$  a.e.  $\mu$ . Since  $\nu(B_n) \leq (n+1)\mu(B_n) < \infty$ , we have  $\chi_{B_n} \in L^p(\nu)$  and by our assumption  $\int_{B_n} f d\mu = 0$ . Therefore by Fatou's lemma we get that  $\int_A g d\mu = 0$ . Thus for all  $A \in \Sigma$  we have  $\int_A g d\mu = 0$ . This means that  $g = 0$  a.e.  $\mu$  and so  $L^p(\nu)$  is dense in  $L^p(\mu)$ .  $\square$

By the Lemma 2.2 we get that  $L^p(\nu)$  is a core of  $M_wEM_u$ . Here we give a condition that we will use in the next theorem.

(★) If  $(\Omega, \mathcal{A}, \mu)$  is a  $\sigma$ -finite measure space and  $J-1 = (E(|u|^q))^{\frac{p}{q}} E(|w|^p) < \infty$  a.e.  $\mu$ , then there exists a sequence  $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$  such that  $\mu(A_n) < \infty$  and  $J-1 < n$  a.e.  $\mu$  on  $A_n$  for every  $n \in \mathbb{N}$  and  $A_n \nearrow \Omega$  as  $n \rightarrow \infty$ .

**Theorem 2.3.** *If  $u, w : \Omega \rightarrow \mathbb{C}$  are  $\Sigma$ -measurable and  $1 < p, q < \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then the following conditions are equivalent:*

- (i)  $M_wEM_u$  is densely defined on  $L^p(\Sigma)$ ,
- (ii)  $J-1 = E(|w|^p)(E(|u|^q))^{\frac{p}{q}} < \infty$  a.e.,  $\mu$ .
- (iii)  $\mu_{J-1} \upharpoonright_{\mathcal{A}}$  is  $\sigma$ -finite.

*Proof.* (i)  $\rightarrow$  (ii) Set  $E = \{E(|w|^p)(E(|u|^q))^{\frac{p}{q}} = \infty\}$ . Clearly by Lemma (i),  $f|_E = 0$  a.e.,  $\mu$  for every  $f \in L^p(\nu)$ . This implies that  $f \cdot J|_E = 0$  a.e.,  $\mu$  for every  $f \in L^p(\mu)$ . So we have  $J \cdot \chi_{A \cap E} = 0$  a.e.,  $\mu$  for all  $A \in \Sigma$  with  $\mu(A) < \infty$ . By the  $\sigma$ -finiteness of  $\mu$  we have  $J \cdot \chi_E = 0$  a.e.,  $\mu$ . Since  $S(J) = \Omega$ , we get that  $\mu(E) = 0$ . (ii)  $\rightarrow$  (i) Evident.

(ii)  $\rightarrow$  (iii) Let  $\{A_n\}_{n=1}^\infty$  be in (★). We have

$$\mu_{J-1} \upharpoonright_{\mathcal{A}} (A_n) = \int_{A_n} E(|w|^p)(E(|u|^q))^{\frac{p}{q}} d\mu \leq n\mu(A_n) < \infty, \quad n \in \mathbb{N}.$$

This yields (iii).

(iii)  $\rightarrow$  (i) Let  $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$  be a sequence such that  $A_n \nearrow \Omega$  as  $n \rightarrow \infty$  and  $\mu_{J-1} \upharpoonright_{\mathcal{A}} (A_n) < \infty$  for every  $k \in \mathbb{N}$ . It follows from the definition of  $\mu_{J-1}$  that  $J-1 = E(|w|^p)(E(|u|^q))^{\frac{p}{q}} < \infty$  a.e.,  $\mu$  on  $\Omega$ . Applying Theorem 2.1, we obtain (i).  $\square$

Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  be a linear operator. If  $T$  is densely defined, then there is a unique maximal operator  $T^*$  from  $\mathcal{D}(T^*) \subset Y^*$  into  $X^*$  such that

$$y^*(Tx) = \langle Tx, y^* \rangle = \langle x, T^*y^* \rangle = T^*y^*(x), \quad x \in \mathcal{D}(T), \quad y^* \in \mathcal{D}(T^*).$$

$T^*$  is called the adjoint of  $T$ .

By Riesz representation theorem for  $L^p$ -spaces we have  $\langle f, F \rangle = F(f) = \int_{\Omega} f \bar{F} d\mu$ , when  $f \in L^p(\Sigma)$ ,  $F \in L^q(\Sigma) = (L^p(\Sigma))^*$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . By the Theorem 2.3 easily we get that: the operator  $M_wEM_u$  is densely defined if and only if the operator  $M_{\bar{u}}EM_{\bar{w}}$  is densely defined. In the next proposition we obtain the adjoint of the WCT operator  $M_wEM_u$  on the Banach space  $L^p(\Sigma)$ .

**Proposition 2.4.** *If the linear transformation  $T = M_wEM_u$  is densely defined on  $L^p(\Sigma)$ , then  $M_{\bar{u}}EM_{\bar{w}}$  is a densely defined operators on  $L^q(\Sigma)$  and  $T^* = M_{\bar{u}}EM_{\bar{w}}$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .*

*Proof.* Let  $f \in \mathcal{D}(T)$  and  $g \in \mathcal{D}(T^*)$ . So we have

$$\begin{aligned}\langle Tf, g \rangle &= \int_{\Omega} wE(uf)\bar{g}d\mu \\ &= \int_{\Omega} fuE(w\bar{g})d\mu \\ &= \langle f, M_{\bar{u}}EM_{\bar{w}}g \rangle.\end{aligned}$$

Hence  $T^* = M_{\bar{u}}EM_{\bar{w}}$ .  $\square$

Now we prove that every densely defined WCT operator is closed.

**Proposition 2.5.** *If  $(E(|u|^q))^{\frac{p}{q}}E(|w|^p) < \infty$  a.e.,  $\mu$ . Then the linear transformation  $M_wEM_u : \mathcal{D}(M_wEM_u) \rightarrow L^p(\Sigma)$  is closed.*

*Proof.* Assume that  $f_n \in \mathcal{D}(M_wEM_u)$ ,  $f_n \rightarrow f$ ,  $wE(uf_n) \rightarrow g$ , and let  $h \in \mathcal{D}(M_{\bar{u}}EM_{\bar{w}})$ . Then

$$\begin{aligned}\langle f, M_{\bar{u}}EM_{\bar{w}}h \rangle &= \lim_{n \rightarrow \infty} \langle f_n, M_{\bar{u}}EM_{\bar{w}}h \rangle \\ &= \lim_{n \rightarrow \infty} \langle wE(uf_n), h \rangle = \langle g, h \rangle.\end{aligned}$$

This calculation (which uses the continuity of the inner product and the fact that  $f_n \in \mathcal{D}(M_wEM_u)$ ) shows that  $f \in \mathcal{D}(M_wEM_u)$  and  $wE(uf) = g$ , as required.  $\square$

In the next theorem we get that if WCT operator  $M_wEM_u$  is densely defined, then it is continuous if and only if it is every where defined.

**Theorem 2.6.** *If  $(E(|u|^q))^{\frac{p}{q}}E(|w|^p) < \infty$  a.e.,  $\mu$ . Then the WCT operator  $M_wEM_u : \mathcal{D}(M_wEM_u) \rightarrow L^p(\Sigma)$  is continuous if and only if it is every where defined i.e.,  $\mathcal{D}(M_wEM_u) = L^p(\Sigma)$ .*

*Proof.* Let  $M_wEM_u$  be continuous. By Lemma 2.2 it is closed. Hence easily we get that  $\mathcal{D}(M_wEM_u)$  is closed and so  $\mathcal{D}(M_wEM_u) = L^p(\Sigma)$ . The converse is easy by closed graph theorem.  $\square$

We denote the range of the operator  $T$  as  $\mathcal{R}(T)$  i.e.,  $\mathcal{R}(T) = \{T(x) : x \in \mathcal{D}(T)\}$ .

**Proposition 2.7.** *If  $E(|u|^2)E(|w|^2) < \infty$  a.e.,  $\mu$  and  $M_wEM_u : \mathcal{D}(M_wEM_u) \subset L^2(\Sigma) \rightarrow L^2(\Sigma)$ , then  $\mathcal{R}(M_wEM_u)$  is closed if and only if  $\mathcal{R}(M_{\bar{u}}EM_{\bar{w}})$  is closed.*

*Proof.* Let  $P : L^2(\Sigma) \times L^2(\Sigma) \rightarrow \mathcal{G}(M_wEM_u)$  be a projection and  $Q : L^2(\Sigma) \times L^2(\Sigma) \rightarrow \{0\} \times L^2(\Sigma)$  be the canonical projection. It is clear that  $\mathcal{R}(M_wEM_u) \cong \mathcal{R}(QP)$ . Also,  $\mathcal{R}(M_{\bar{u}}EM_{\bar{w}}) \cong \mathcal{R}((I - Q)(I - P))$ . Since  $P$  and  $Q$  are orthogonal projections, then  $\mathcal{R}(QP)$  is closed if and only if  $\mathcal{R}((I - Q)(I - P))$ . Thus we obtain the desired result.  $\square$

It is well-known that for a densely defined closed operator  $T$  of  $\mathcal{H}_1$  into  $\mathcal{H}_2$ , there exists a partial isometry  $U_T$  with initial space  $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(|T|)}$  and final space  $\mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}$  such that

$$T = U_T|T|.$$

**Theorem 2.8.** *Suppose that  $\mathcal{D}(M_w EM_u)$  is dense in  $L^2(\Sigma)$ . Let  $M_w EM_u = U|M_w EM_u|$  be the polar decomposition of  $M_w EM_u$ . Then*

$$(i) \quad |M_w EM_u| = M_{u'} EM_u, \text{ where } u' = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \cdot \chi_{S \cap \bar{u}} \text{ and } S = S(E(|u|^2)),$$

(ii)  $U = M_{w'} EM_u$ , where  $w' : \Omega \rightarrow \mathbb{C}$  is an a.e.  $\mu$  well-defined  $\Sigma$ -measurable function such that

$$w' = \frac{w}{(E(|w|^2)E(|u|^2))^{\frac{1}{2}}} \cdot \chi_{S \cap G},$$

in which  $G = S(E(|w|^2))$ .

*Proof.* (i). For every  $f \in \mathcal{D}(M_{u'} EM_u)$  we have

$$\|M_{u'} EM_u(f)(f)\|^2 = \| |M_w EM_u|(f) \|^2.$$

Also, by Lemma 2.2 we conclude that  $\mathcal{D}(M_{u'} EM_u) = \mathcal{D}(|M_w EM_u|)$  and it is easily seen that  $M_{u'} EM_u$  is a positive operator. These observations imply that  $|M_w EM_u| = M_{u'} EM_u$ .

(ii). For  $f \in L^2(\Sigma)$  we have

$$\int_{\Omega} |w' E(uf)|^2 d\mu = \int_{\Omega} \frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)} |w E(uf)|^2 d\mu,$$

which implies that the operator  $M_{w'} EM_u$  is well-defined and  $\mathcal{N}(M_w EM_u) = \mathcal{N}(M_{w'} EM_u)$ . Also, for  $f \in \mathcal{D}(M_w EM_u) \ominus \mathcal{N}(M_w EM_u)$  we have

$$U(|M_w EM_u|(f)) = w E(uf) \cdot \chi_{S \cap G} = w E(uf).$$

Thus  $\|U(f)\| = \|f\|$  for all  $f \in \mathcal{R}(|M_w EM_u|)$  and since  $U$  is a contraction, then it holds for all  $f \in \mathcal{N}(M_w EM_u)^\perp = \mathcal{R}(|M_w EM_u|)$ .  $\square$

Here we remind that: if  $T : \mathcal{D}(T) \subset X \rightarrow X$  is a closed linear operator on the Banach space  $X$ , then a complex number  $\lambda$  belongs to the resolvent set  $\rho(T)$  of  $T$ , if the operator  $\lambda I - T$  has a bounded everywhere on  $X$  defined inverse  $(\lambda I - T)^{-1}$ , called the resolvent of  $T$  at  $\lambda$  and denoted by  $R_\lambda(T)$ . The set  $\sigma(T) := \mathbb{C} \setminus \rho(T)$  is called the spectrum of the operator  $T$ .

It is known that, if  $a, b$  are elements of a unital algebra  $A$ , then  $1 - ab$  is invertible if and only if  $1 - ba$  is invertible. A consequence of this equivalence is that  $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ . Now, in the next theorem we compute the spectrum of WCT operator  $M_w EM_u$  as a densely defined operator on  $L^2(\Sigma)$ .

**Proposition 2.9.** *Let  $M_w EM_u$  be densely defined and  $\mathcal{A} \subsetneq \Sigma$ , then*

$$(i) \quad \text{essrange}(E(uw)) \setminus \{0\} \subseteq \sigma(M_w EM_u),$$

$$(ii) \quad \text{If } L^2(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw}), \text{ then } \sigma(M_w EM_u) \setminus \{0\} \subseteq \text{essrange}(E(uw)) \setminus \{0\}.$$

*Proof.* Since  $\sigma(M_w EM_u) \setminus \{0\} = \sigma(EM_{uw}) \setminus \{0\}$ , then by using theorem 2.8 of [3] we get the proof.  $\square$

By a similar method that we used in the proof of theorem 2.8 of [3] we have the same assertion for the spectrum of the densely defined operator  $EM_u$  on the space  $L^p(\Sigma)$ , i.e.,

$$(i) \quad \text{essrange}(E(u)) \cup \{0\} \subseteq \sigma(EM_u),$$

(ii) If  $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_u)$ , then  $\sigma(EM_u) \subseteq \text{essrange}(E(u)) \cup \{0\}$ .

By these observations we have the next remark.

*Remark 2.10.* Let  $M_w EM_u$  be densely defined operator on  $L^p(\Sigma)$  and  $\mathcal{A} \subsetneq \Sigma$ , then

(i)  $\text{essrange}(E(uw)) \setminus \{0\} \subseteq \sigma(M_w EM_u)$ ,

(ii) If  $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw})$ , then  $\sigma(M_w EM_u) \setminus \{0\} \subseteq \text{essrange}(E(uw)) \setminus \{0\}$ .

As we know the spectral radius of a densely defined operator  $T$  is denoted by  $r(T)$  and is defined as:  $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$ . Hence we have the next corollary.

**Corollary 2.11.** *If the WCT operator  $M_w EM_u$  is densely defined on  $L^p(\Sigma)$  and  $L^p(\mathcal{A}) \subseteq \mathcal{D}(EM_{uw})$ , then  $\sigma(M_w EM_u) \setminus \{0\} = \text{essrange}(E(uw)) \setminus \{0\}$  and  $r(M_w EM_u) = \|E(uw)\|_\infty$ .*

A densely defined operator  $T$  on the Hilbert space  $\mathcal{H}$  is said to be *hyponormal* if  $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$  and  $\|T^*(f)\| \leq \|T(f)\|$  for  $f \in \mathcal{D}(T)$ . A densely defined operator  $T$  on the Hilbert space  $\mathcal{H}$  is said to be *normal* if  $T$  is closed and  $T^*T = TT^*$ . For the WCT operator  $T = M_w EM_u$  on  $L^2(\Sigma)$  we have  $T^* = M_{\bar{u}} EM_{\bar{w}}$  and we recall that  $T$  is densely defined if and only if  $T^*$  is densely defined. If  $T$  is densely defined, then by the Lemma 2.2 we get that  $L^2(\nu) \subseteq \mathcal{D}(T)$ ,  $L^2(\nu) \subseteq \mathcal{D}(T^*)$  and

$$\overline{L^2(\nu)}^{\|\cdot\|_\mu} = \overline{\mathcal{D}(T)}^{\|\cdot\|_\mu} = \overline{\mathcal{D}(T^*)}^{\|\cdot\|_\mu} = L^2(\mu),$$

in which  $d\nu = Jd\mu$  and  $J = 1 + E(|w|^2)E(|u|^2)$ . Also, we have  $T^*T = M_{E(|w|^2)\bar{u}} EM_u$  and  $TT^* = M_{E(|u|^2)w} EM_{\bar{w}}$ . Similarly, we have  $L^2(\nu') \subseteq \mathcal{D}(T^*T)$ ,  $L^2(\nu') \subseteq \mathcal{D}(TT^*)$  and

$$\overline{L^2(\nu')}^{\|\cdot\|_\mu} = \overline{\mathcal{D}(T^*T)}^{\|\cdot\|_\mu} = \overline{\mathcal{D}(TT^*)}^{\|\cdot\|_\mu} = L^2(\mu),$$

in which  $d\nu' = J'd\mu$  and  $J' = 1 + (E(|w|^2))^2(E(|u|^2))^2$ . By these observations we have next assertions.

**Proposition 2.12.** *Let WCT operator  $M_w EM_u$  be densely defined on  $L^2(\Sigma)$ . Then we have the followings:*

(i) *If  $u(E(|w|^2))^{\frac{1}{2}} = \bar{w}(E(|u|^2))^{\frac{1}{2}}$  with respect to the measure  $\mu$ , then  $T = M_w EM_u$  is normal.*

(ii) *If  $T = M_w EM_u$  is normal, then  $E(|w|^2)|E(u)|^2 = E(|u|^2)|E(w)|^2$  with respect to the measure  $\mu$ .*

*Proof.* (i) Direct computations shows that

$$T^*T - TT^* = M_{\bar{u}E(|w|^2)} EM_u - M_{wE(|u|^2)} EM_{\bar{w}},$$

on  $L^2(\nu')$ . Hence for every  $f \in L^2(\nu')$  we have

$$\begin{aligned} \langle T^*T - TT^*(f), f \rangle &= \int_X E(|w|^2)E(uf)\bar{u}f - E(|u|^2)E(\bar{w}f)w\bar{f}d\mu \\ &= \int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2d\mu. \end{aligned}$$

This implies that if

$$(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}},$$

then for all  $f \in L^2(\nu')$ ,  $\langle T^*T - TT^*(f), f \rangle = 0$ . Thus  $T^*T = TT^*$ .

(ii) Suppose that  $T$  is normal. By (i), for all  $f \in L^2(\nu')$  we have

$$\int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2 d\mu = 0.$$

Let  $A \in \mathcal{A}$ , with  $0 < \nu'(A) < \infty$ . By replacing  $f$  to  $\chi_A$ , we have

$$\int_A |E(u(E(|w|^2))^{\frac{1}{2}})|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w})|^2 d\mu = 0$$

and so

$$\int_A |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) d\mu = 0.$$

Since  $A \in \mathcal{A}$  is arbitrary and  $\mu \ll \nu'$  (absolutely continuous), then  $|E(u)|^2 E(|w|^2) = |E(w)|^2 E(|u|^2)$  with respect to  $\mu$ .  $\square$

**Proposition 2.13.** *Let the WCT operator  $M_w EM_u$  be densely defined on  $L^2(\Sigma)$ . Then we have the followings:*

(i) *If  $u(E(|w|^2))^{\frac{1}{2}} \geq \bar{w}(E(|u|^2))^{\frac{1}{2}}$  with respect to  $\mu$ , then  $T = M_w EM_u$  is hyponormal.*

(ii) *If  $T = M_w EM_u$  is hyponormal, then  $E(|w|^2)|E(u)|^2 \geq E(|u|^2)|E(w)|^2$  with respect to the measure  $\mu$ .*

*Proof.* By a similar method of 2.12 we can get the proof.  $\square$

### 3. Hyperexpansive WCT operators

In this section we are going to present conditions under which WCT operator  $M_w EM_u$  on  $L^2(\Sigma)$  is  $k$ -isometry,  $k$ -expansive,  $k$ -hyperexpansive and completely hyperexpansive. For an operator  $T$  on the Hilbert space  $\mathcal{H}$  we set

$$\Theta_{T,n}(f) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \|T^i(f)\|^2, \quad f \in \mathcal{D}(T^n), \quad n \geq 1.$$

By means of this definition an operator  $T$  on  $\mathcal{H}$  is said to be:

- (i)  $k$ -isometry ( $k \geq 1$ ) if  $\Theta_{T,k}(f) = 0$  for  $f \in \mathcal{D}(T^k)$ ,
- (ii)  $k$ -expansive ( $k \geq 1$ ) if  $\Theta_{T,k}(f) \leq 0$  for  $f \in \mathcal{D}(T^k)$ ,
- (iii)  $k$ -hyperexpansive ( $k \geq 1$ ) if  $\Theta_{T,n}(f) \leq 0$  for  $f \in \mathcal{D}(T^n)$  and  $n = 1, 2, \dots, k$ .
- (iv) completely hyperexpansive if  $\Theta_{T,n}(f) \leq 0$  for  $f \in \mathcal{D}(T^n)$  and  $n \geq 1$ .

For more details one can see [6, 7, 11]. It is easily seen that for every  $f \in L^2(\Sigma)$

$$\|M_w EM_u(f)\|_2 = \|EM_v(f)\|_2,$$

where  $v = u(E(|w|^2))^{\frac{1}{2}}$ . Thus without loss of generality we can consider the operator  $EM_v$  instead of  $M_w EM_u$  in our discussion. First we recall some concepts that we need them in the sequel. Now we present our main results. The next lemma is a direct consequence of Theorem 2.3.



**Lemma 3.1.** *For every  $n \in \mathbb{N}$  the operator  $(EM_v)^n$  on  $L^2(\Sigma)$  is densely-defined if and only if the operator  $EM_v$  is densely defined on  $L^2(\Sigma)$ .*

In the Theorem 3.2 we give some necessary and sufficient conditions for  $k$ -isometry and  $k$ -expansive WCT operators  $EM_v$ .

**Theorem 3.2.** *If  $\mathcal{D}(EM_v)$  is dense in  $L^2(\mu)$ , then:*

(i) *If the operator  $EM_v$  is  $k$ -isometry ( $k \geq 1$ ), then  $A_k^0(|E(v)|^2) = 0$ ;*

(ii) *If  $(1 + E(|v|^2)A_k^1(|E(v)|^2)) = 0$  and  $|E(vf)|^2 = E(|v|^2)E(|f|^2)$  for all  $f \in \mathcal{D}(EM_v)$ , then the operator  $EM_v$  is  $k$ -isometry;*

(iii) *If the operator  $EM_v$  is  $k$ -expansive, then  $A_k^0(|E(v)|^2) \leq 0$ ;*

(iv) *If  $(1 + E(|v|^2)A_k^1(|E(v)|^2)) \leq 0$  and  $|E(vf)|^2 = E(|v|^2)E(|f|^2)$  for all  $f \in \mathcal{D}(EM_v)$ , then the operator  $EM_v$  is  $k$ -expansive, where*

$$A_k^0(|E(v)|^2) = \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} |E(v)|^{2i}, \quad A_k^1(|E(v)|^2) = \sum_{1 \leq i \leq k} (-1)^i \binom{k}{i} |E(v)|^{2(i-1)}.$$

*Proof.* Suppose that the operator  $EM_v$  is  $k$ -isometry. So for all  $f \in \mathcal{D}((EM_v)^k)$  we have

$$\begin{aligned} 0 &= \Theta_{T,k}(f) \\ &= \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} \|(EM_v)^i(f)\|^2 \\ &= \int_{\Omega} |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_{\Omega} |E(v)|^{2(i-1)} |E(vf)|^2 d\mu, \end{aligned}$$

and so for all  $\mathcal{A}$ -measurable functions  $f \in \mathcal{D}((EM_v)^k)$

$$\begin{aligned} 0 &= \int_{\Omega} |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_{\Omega} |E(v)|^{2(i-1)} |E(v)|^2 |f|^2 d\mu \\ &= \int_{\Omega} \left( \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} |E(v)|^{2i} \right) |f|^2 d\mu. \end{aligned}$$

Since  $(EM_v)^k$  is densely defined, then we get that  $A_k(|E(v)|^2) = 0$ .

(ii) Let  $1 + E(|v|^2)A_k^1(|E(v)|^2) = 0$  and  $|E(vf)|^2 = E(|v|^2)E(|f|^2)$  for all  $f \in$

$\mathcal{D}((EM_v)^k)$ . Then for all  $f \in \mathcal{D}((EM_v)^k)$  we have

$$\begin{aligned}
\Theta_{T,k}(f) &= \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} \|(EM_v)^i(f)\|^2 \\
&= \int_{\Omega} |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_{\Omega} |E(v)|^{2(i-1)} |E(vf)|^2 d\mu \\
&= \int_{\Omega} |f|^2 d\mu + \int_{\Omega} \left( \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} (E(|v|^2))^{2(i-1)} \right) E(|v|^2) E(|f|^2) d\mu \\
&= \int_{\Omega} (1 + E(|v|^2) A_k(|E(v)|^2)) |f|^2 d\mu \\
&= 0.
\end{aligned}$$

This implies that the operator  $EM_v$  is  $k$ -isometry.

(iii), (iv). By the same method that is used in (i) and (ii), easily we get (iii) and (iv). □

Here we recall that if the linear transformation  $T = EM_v$  is densely defined on  $L^2(\Sigma)$ , then  $T = EM_v$  is closed and  $T^* = M_{\bar{v}}E$ . Also, if  $\mathcal{D}(EM_v)$  is dense in  $L^2(\Sigma)$  and  $v$  is almost every where finite valued, then the operator  $EM_v$  is normal if and only if  $v \in L^0(\mathcal{A})$  [3]. Hence we have the Remark 3.3 for normal WCT operators.

*Remark 3.3.* Suppose that the operator  $EM_v$  is normal and  $\mathcal{D}(EM_v)$  is dense in  $L^2(\mu)$  for a fixed  $k \geq 1$ . If  $|E(f)|^2 = E(|f|^2)$  on  $S(v)$  for all  $f \in \mathcal{D}((EM_v)^k)$ , then:

- (i) The operator  $EM_v$  is  $k$ -isometry ( $k \geq 1$ ) if and only if  $A_k(|v|^2) = 0$ ;
- (ii) The operator  $EM_v$  is  $k$ -expansive if and only if  $A_k(|v|^2) \leq 0$ .

*Proof.* Since  $EM_v$  is normal, then  $|E(v)|^2 = E(|v|^2) = |v|^2$ . Thus by Theorem 3.2 we have (i) and (ii). □

Here we give some properties of 2-expansive WCT operators and as a corollary for 2-expansive multiplication operators.

**Proposition 3.4.** *If  $\mathcal{D}(EM_v)$  is dense in  $L^2(\mu)$  and  $EM_v$  is 2-expansive, then:*

- (i)  $EM_v$  leaves its domain invariant:
- (ii)  $|E(v)|^{2k} \geq |E(v)|^{2(k-1)}$  a.e.  $\mu$  for all  $k \geq 1$ .

*Proof.* (i). Since  $EM_v$  is 2-expansive, we get that for every  $f \in \mathcal{D}(EM_v)$

$$\begin{aligned} \|(EM_v)^2(f)\|^2 &= \int_{\Omega} |E(v)|^2 |E(vf)|^2 d\mu \\ &\leq 2 \int_{\Omega} |E(vf)|^2 d\mu - \int_{\Omega} |f|^2 d\mu \\ &< \infty, \end{aligned}$$

so  $EM_v(f) \in \mathcal{D}(EM_v)$ .

(ii) Since  $EM_v$  leaves its domain invariant, then  $\mathcal{D}(EM_v) \subseteq \mathcal{D}^\infty(EM_v)$ . So by lemma 3.2 (iii) of [7] we get that  $\|(EM_v)^k(f)\|^2 \geq \|(EM_v)^{k-1}(f)\|^2$  for all  $f \in \mathcal{D}(EM_v)$  and  $k \geq 1$  we have

$$\int_{\Omega} |E(v)|^{2(k-1)} |E(vf)|^2 d\mu \geq \int_{\Omega} |E(v)|^{2(k-2)} |E(vf)|^2 d\mu,$$

and so

$$\int_{\Omega} (|E(v)|^{2(k-1)} - |E(v)|^{2(k-2)}) |E(vf)|^2 d\mu \geq 0,$$

for all  $f \in \mathcal{D}(EM_v)$ . This leads to  $|E(v)|^{2k} \geq |E(v)|^{2(k-1)}$  a.e.,  $\mu$ .  $\square$

**Corollary 3.5.** *If  $\mathcal{D}(M_v)$  is dense in  $L^2(\mu)$  and  $M_v$  is 2-expansive, then:*

- (i)  $M_v$  leaves its domain invariant:
- (ii)  $v^{2k} \geq v^{2(k-1)}$  a.e.  $\mu$  for all  $k \geq 1$ .

Recall that a real-valued map  $\phi$  on  $\mathbb{N}$  is said to be completely alternating if  $\sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \phi(m+i) \leq 0$  for all  $m \geq 0$  and  $n \geq 1$ . The next remark is a direct consequence of Lemma 3.1 and Theorem 3.2.

*Remark 3.6.* If  $\mathcal{D}(EM_v)$  is dense in  $L^2(\mu)$  and  $k \geq 1$  is fixed, then:

- (i) If the operator  $EM_v$  is  $k$ -hyperexpansive ( $k \geq 1$ ), then  $A_n^0(|E(v)|^2) \leq 0$  for  $n = 1, 2, \dots, k$ ;
- (ii) If  $(1 + E(|v|^2)A_n^1(|E(v)|^2)) \leq 0$  and  $|E(vf)|^2 = E(|v|^2)E(|f|^2)$  for all  $f \in \mathcal{D}(EM_v)^n$  and  $n = 1, 2, \dots, k$ , then the operator  $EM_v$  is  $k$ -hyperexpansive ( $k \geq 1$ );
- (iii) If the operator  $EM_v$  is completely hyperexpansive, then
  - (a) the sequence  $\{|E(v)(t)|^2\}_{n=0}^\infty$  is a completely alternating sequence for almost every  $t \in \Omega$ ,
  - (b)  $A_n^0(|E(v)|^2) \leq 0$  for  $n \geq 1$ .
- (iv) If  $(1 + E(|v|^2)A_n^1(|E(v)|^2)) \leq 0$  and  $|E(vf)|^2 = E(|v|^2)E(|f|^2)$  for all  $f \in \mathcal{D}((EM_v)^n)$  and  $n \geq 1$ , then the operator  $EM_v$  is completely hyperexpansive.

By Remark 3.6 and some properties of normal WCT operators we get the next remark for  $k$ -hyperexpansive and completely hyperexpansive normal WCT operators.

*Remark 3.7.* Let the operator  $EM_v$  be normal,  $\mathcal{D}(EM_v)$  be dense in  $L^2(\mu)$  and  $k \geq 1$  be fixed. If  $|E(f)|^2 = E(|f|^2)$  on  $S(v)$  for all  $f \in \mathcal{D}((EM_v)^k)$ , then

(i)  $EM_v$  is  $k$ -hyperexpansive ( $k \geq 1$ ) if and only if  $A_n(|v|^2) \leq 0$  for  $f \in \mathcal{D}(T^n)$  and  $n = 1, 2, \dots, k$ .

(ii)  $EM_v$  is completely hyperexpansive if and only if the sequence  $\{|u(t)|^2\}_{n=0}^\infty$  is a completely alternating sequence for almost every  $t \in \Omega$ ,

If all functions  $v^{2^i}$  for  $i = 1, \dots, n$  are finite valued, then we set

$$\Delta_{v,n}(x) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} |v|^{2^i}(t).$$

Also, if  $\mathcal{A} = \Sigma$ , then  $E = I$ . So we have next two corollaries.

**Corollary 3.8.** *If  $\mathcal{D}(M_v)$  is dense in  $L^2(\mu)$  for a fixed  $n \geq 1$ , then:*

(i)  $M_v$  is  $k$ -expansive if and only if  $\Delta_{v,n}(x) \leq 0$  a.e.  $\mu$ .

(ii)  $M_v$  is  $k$ -isometry if and only if  $\Delta_{v,n}(x) = 0$  a.e.  $\mu$ .

**Corollary 3.9.** *Let  $\mathcal{D}(M_v)$  be dense in  $L^2(\mu)$  and  $k \geq 1$  be fixed. Then*

(i)  $M_v$  is  $k$ -hyperexpansive ( $k \geq 1$ ) if and only if  $\Delta_{v,n}(t) \leq 0$  a.e.,  $\mu$  for  $n = 1, 2, \dots, k$ .

(ii)  $M_v$  is completely hyperexpansive if and only if the sequence  $\{|u(t)|^2\}_{n=0}^\infty$  is a completely alternating sequence for almost every  $t \in \Omega$ .

Finally we give some examples.

**Example 3.10.** *Let  $\Omega = [-1, 1]$ ,  $d\mu = \frac{1}{2}dx$  and  $\mathcal{A} = \langle \{(-a, a) : 0 \leq a \leq 1\} \rangle$  (Sigma algebra generated by symmetric intervals). Then*

$$E^{\mathcal{A}}(f)(t) = \frac{f(t) + f(-t)}{2}, \quad t \in \Omega,$$

where  $E^{\mathcal{A}}(f)$  is defined. If  $v(t) = e^t$ , then  $E^{\mathcal{A}}(v)(t) = \cosh(t)$  and we have the followings:

1)  $E^{\mathcal{A}}M_v$  is densely defined and closed on  $L^p(\Omega)$ .

2)  $\sigma(E^{\mathcal{A}}M_v) = \mathcal{R}(\cosh(t))$ .

3)  $E^{\mathcal{A}}M_v$  is not 2-expansive, since

$$\begin{aligned} 1 - 2|E(v)|^2(t) + |E(v)|^4(t) &= 1 - 2\cosh^2(t) + \cosh^4(t) \\ &= (\cosh^2(t) - 1)^2 \geq 0. \end{aligned}$$

**Example 3.11.** Let  $\Omega = \mathbb{N}$ ,  $\mathcal{G} = 2^{\mathbb{N}}$  and let  $\mu(\{t\}) = pq^{t-1}$ , for each  $t \in \Omega$ ,  $0 \leq p \leq 1$  and  $q = 1 - p$ . Elementary calculations show that  $\mu$  is a probability measure on  $\mathcal{G}$ . Let  $\mathcal{A}$  be the  $\sigma$ -algebra generated by the partition  $B = \{\Omega_1 = \{3n : n \geq 1\}, \Omega_1^c\}$  of  $\Omega$ . So, for every  $f \in \mathcal{D}(E^{\mathcal{A}})$  we have

$$E(f) = \alpha_1 \chi_{\Omega_1} + \alpha_2 \chi_{\Omega_1^c}$$

and direct computations show that

$$\alpha_1(f) = \frac{\sum_{n \geq 1} f(3n) pq^{3n-1}}{\sum_{n \geq 1} pq^{3n-1}}$$

and

$$\alpha_2(f) = \frac{\sum_{n \geq 1} f(n) pq^{n-1} - \sum_{n \geq 1} f(3n) pq^{3n-1}}{\sum_{n \geq 1} pq^{n-1} - \sum_{n \geq 1} pq^{3n-1}}.$$

So, if  $u$  and  $w$  are real functions on  $\Omega$ . Then we have the followings:

1) If  $\alpha_1((|u|^q)^{\frac{p}{q}}) \alpha_1(|w|^p) < \infty$  and  $\alpha_2((|u|^q)^{\frac{p}{q}}) \alpha_2(|w|^p) < \infty$ , then the operator  $M_w EM_u$  is a densely defined and closed operator on  $L^p(\Omega)$ .

2)  $\sigma(M_w EM_u) = \{\alpha_1(E(uw)), \alpha_2(E(uw))\}$ .

**Example 3.12.** Let  $\Omega = [0, 1] \times [0, 1]$ ,  $d\mu = dt dt'$ ,  $\Sigma$  the Lebesgue subsets of  $\Omega$  and let  $\mathcal{A} = \{A \times [0, 1] : A \text{ is a Lebesgue set in } [0, 1]\}$ . Then, for each  $f$  in  $L^2(\Sigma)$ ,  $(Ef)(t, t') = \int_0^1 f(t, s) ds$ , which is independent of the second coordinate. Hence for  $v(t, t') = t^m$  we get that  $v$  is  $\mathcal{A}$ -measurable and  $EM_v$  is  $k$ -expansive and  $k$ -isometry if

$$\sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} x^{2mi} \leq 0, \quad \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} t^{2mi} = 0,$$

respectively. This example is valid in the general case as follows:

Let  $(\Omega_1, \Sigma_1, \mu_1)$  and  $(\Omega_2, \Sigma_2, \mu_2)$  be two  $\sigma$ -finite measure spaces and  $\Omega = \Omega_1 \times \Omega_2$ ,  $\Sigma = \Sigma_1 \times \Sigma_2$  and  $\mu = \mu_1 \times \mu_2$ . Put  $\mathcal{A} = \{A \times \Omega_2 : A \in \Sigma_1\}$ . Then  $\mathcal{A}$  is a sub- $\sigma$ -algebra of  $\Sigma$ . Then for all  $f$  in domain  $E^{\mathcal{A}}$  we have

$$E^{\mathcal{A}}(f)(t_1) = E^{\mathcal{A}}(f)(t_1, t_2) = \int_{\Omega_2} f(t_1, s) d\mu_2(s) \quad \mu - a.e.$$

on  $\Omega$ .

Also, if  $(\Omega, \Sigma, \mu)$  is a finite measure space and  $k : \Omega \times \Omega \rightarrow \mathbb{C}$  is a  $\Sigma \otimes \Sigma$ -measurable function such that

$$\int_{\Omega} |k(., s) f(s)| d\mu(s) \in L^2(\Sigma)$$

for all  $f \in L^2(\Sigma)$ . Then the operator  $T : L^2(\Sigma) \rightarrow L^2(\Sigma)$  defined by

$$Tf(t) = \int_{\Omega} k(t, s) f(s) d\mu, \quad f \in L^2(\Sigma),$$

is called kernel operator on  $L^2(\Sigma)$ . We show that  $T$  is a weighted conditional type operator.[5] Since  $L^2(\Sigma) \times \{1\} \cong L^2(\Sigma)$  and  $vf$  is a  $\Sigma \otimes \Sigma$ -measurable function,

when  $f \in L^2(\Sigma)$ . Then by taking  $v := k$  and  $f'(t, s) = f(s)$ , we get that

$$\begin{aligned} E^{\mathcal{A}}(vf)(t) &= E^{\mathcal{A}}(vf')(t, s) \\ &= \int_{\Omega} v(t, t')f'(t', s)d\mu(t') \\ &= \int_{\Omega} v(t, t')f(t')d\mu(t') \\ &= Tf(t). \end{aligned}$$

Hence  $T = EM_v$ , i.e,  $T$  is a weighted conditional type operator. This means all assertions of this paper are valid for a class of integral type operators.

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